14 | INDUCTANCE

Figure 14.1 A smartphone charging mat contains a coil that receives alternating current, or current that is constantly increasing and decreasing. The varying current induces an emf in the smartphone, which charges its battery. Note that the black box containing the electrical plug also contains a transformer (discussed in **Alternating-Current Circuits**) that modifies the current from the outlet to suit the needs of the smartphone. (credit: modification of work by "LG"/Flickr)

Chapter Outline

Introduction

In **Electromagnetic Induction**, we discussed how a time-varying magnetic flux induces an emf in a circuit. In many of our calculations, this flux was due to an applied time-dependent magnetic field. The reverse of this phenomenon also occurs: The current flowing in a circuit produces its own magnetic field.

In **Electric Charges and Fields**, we saw that induction is the process by which an emf is induced by changing magnetic flux. So far, we have discussed some examples of induction, although some of these applications are more effective than others. The smartphone charging mat in the chapter opener photo also works by induction. Is there a useful physical quantity related to how "effective" a given device is? The answer is yes, and that physical quantity is *inductance*. In this chapter, we look at the applications of inductance in electronic devices and how inductors are used in circuits.

14.1 | Mutual Inductance

Learning Objectives

By the end of this section, you will be able to:

- Correlate two nearby circuits that carry time-varying currents with the emf induced in each circuit
- Describe examples in which mutual inductance may or may not be desirable

Inductance is the property of a device that tells us how effectively it induces an emf in another device. In other words, it is a physical quantity that expresses the effectiveness of a given device.

When two circuits carrying time-varying currents are close to one another, the magnetic flux through each circuit varies because of the changing current I in the other circuit. Consequently, an emf is induced in each circuit by the changing current in the other. This type of emf is therefore called a *mutually induced emf*, and the phenomenon that occurs is known as **mutual inductance (*M*)**. As an example, let's consider two tightly wound coils (Figure 14.2). Coils 1 and 2 have N_1 and N_2 turns and carry currents I_1 and $I_2,$ respectively. The flux through a single turn of coil 2 produced by the magnetic field of the current in coil 1 is $\Phi_{21},$ whereas the flux through a single turn of coil 1 due to the magnetic field of I_2 is $\Phi_{12}.$

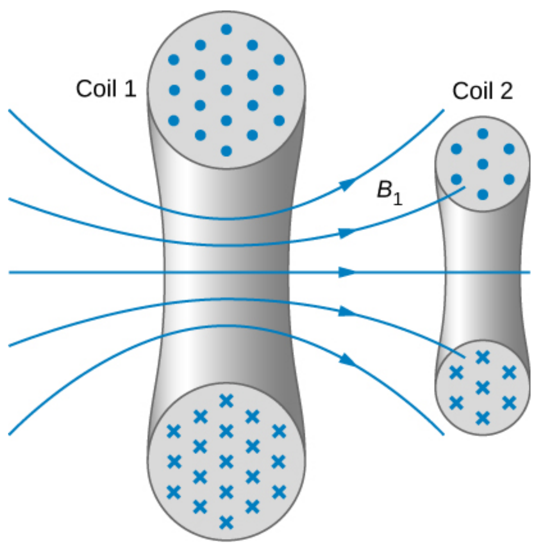


Figure 14.2 Some of the magnetic field lines produced by the current in coil 1 pass through coil 2.

The mutual inductance M_{21} of coil 2 with respect to coil 1 is the ratio of the flux through the N_2 turns of coil 2 produced by the magnetic field of the current in coil 1, divided by that current, that is,

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1}. \tag{14.1}$$

Similarly, the mutual inductance of coil 1 with respect to coil 2 is

$$M_{12} = \frac{N_1 \Phi_{12}}{I_2}. \tag{14.2}$$

Like capacitance, mutual inductance is a geometric quantity. It depends on the shapes and relative positions of the two coils, and it is independent of the currents in the coils. The SI unit for mutual inductance M is called the **henry (H)** in honor of

Joseph Henry (1799–1878), an American scientist who discovered induced emf independently of Faraday. Thus, we have $1\text{ H} = 1\text{ V} \cdot \text{s/A}$. From **Equation 14.1** and **Equation 14.2**, we can show that $M_{21} = M_{12}$, so we usually drop the subscripts associated with mutual inductance and write

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}. \tag{14.3}$$

The emf developed in either coil is found by combining Faraday's law and the definition of mutual inductance. Since $N_2 \Phi_{21}$ is the total flux through coil 2 due to I_1, we obtain

$$\varepsilon_2 = -\frac{d}{dt}(N_2 \Phi_{21}) = -\frac{d}{dt}(MI_1) = -M\frac{dI_1}{dt} \tag{14.4}$$

where we have used the fact that M is a time-independent constant because the geometry is time-independent. Similarly, we have

$$\varepsilon_1 = -M\frac{dI_2}{dt}. \tag{14.5}$$

In **Equation 14.5**, we can see the significance of the earlier description of mutual inductance (M) as a geometric quantity. The value of M neatly encapsulates the physical properties of circuit elements and allows us to separate the physical layout of the circuit from the dynamic quantities, such as the emf and the current. **Equation 14.5** defines the mutual inductance in terms of properties in the circuit, whereas the previous definition of mutual inductance in **Equation 14.1** is defined in terms of the magnetic flux experienced, regardless of circuit elements. You should be careful when using **Equation 14.4** and **Equation 14.5** because ε_1 and ε_2 do not necessarily represent the total emfs in the respective coils. Each coil can also have an emf induced in it because of its *self-inductance* (self-inductance will be discussed in more detail in a later section).

A large mutual inductance M may or may not be desirable. We want a transformer to have a large mutual inductance. But an appliance, such as an electric clothes dryer, can induce a dangerous emf on its metal case if the mutual inductance between its coils and the case is large. One way to reduce mutual inductance is to counter-wind coils to cancel the magnetic field produced (**Figure 14.3**).

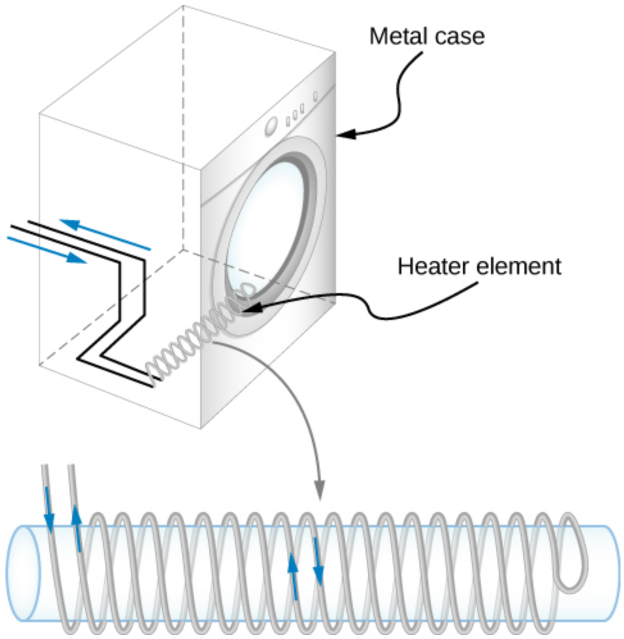


Figure 14.3 The heating coils of an electric clothes dryer can be counter-wound so that their magnetic fields cancel one another, greatly reducing the mutual inductance with the case of the dryer.

Digital signal processing is another example in which mutual inductance is reduced by counter-winding coils. The rapid on/off emf representing 1s and 0s in a digital circuit creates a complex time-dependent magnetic field. An emf can be generated in neighboring conductors. If that conductor is also carrying a digital signal, the induced emf may be large enough to switch 1s and 0s, with consequences ranging from inconvenient to disastrous.

Example 14.1

Mutual Inductance

Figure 14.4 shows a coil of N_2 turns and radius R_2 surrounding a long solenoid of length l_1, radius R_1, and N_1 turns. (a) What is the mutual inductance of the two coils? (b) If $N_1 = 500\,\text{turns}$, $N_2 = 10\,\text{turns}$, $R_1 = 3.10\,\text{cm}$, $l_1 = 75.0\,\text{cm}$, and the current in the solenoid is changing at a rate of 200 A/s, what is the emf induced in the surrounding coil?

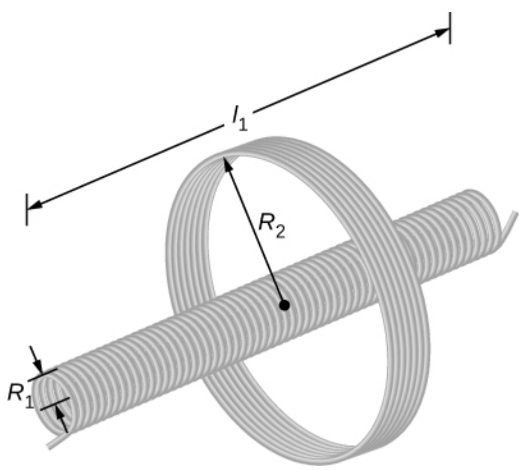


Figure 14.4 A solenoid surrounded by a coil.

Strategy

There is no magnetic field outside the solenoid, and the field inside has magnitude $B_1 = \mu_0 (N_1 / l_1) I_1$ and is directed parallel to the solenoid's axis. We can use this magnetic field to find the magnetic flux through the surrounding coil and then use this flux to calculate the mutual inductance for part (a), using **Equation 14.3**. We solve part (b) by calculating the mutual inductance from the given quantities and using **Equation 14.4** to calculate the induced emf.

Solution

a. The magnetic flux Φ_{21} through the surrounding coil is

$$\Phi_{21} = B_1 \pi R_1^2 = \frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2.$$

Now from **Equation 14.3**, the mutual inductance is

$$M = \frac{N_2 \Phi_{21}}{I_1} = \left(\frac{N_2}{I_1}\right)\left(\frac{\mu_0 N_1 I_1}{l_1}\right)\pi R_1^2 = \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}.$$

b. Using the previous expression and the given values, the mutual inductance is

$$\begin{aligned} M &= \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(500)(10)\pi(0.0310\ \text{m})^2}{0.750\ \text{m}} \\ &= 2.53 \times 10^{-5}\ \text{H}. \end{aligned}$$

Thus, from **Equation 14.4**, the emf induced in the surrounding coil is

$$\begin{aligned} \varepsilon_2 &= -M\frac{dI_1}{dt} = -(2.53 \times 10^{-5}\ \text{H})(200\ \text{A/s}) \\ &= -5.06 \times 10^{-3}\ \text{V}. \end{aligned}$$

Significance

Notice that M in part (a) is independent of the radius R_2 of the surrounding coil because the solenoid's magnetic field is confined to its interior. In principle, we can also calculate M by finding the magnetic flux through the solenoid produced by the current in the surrounding coil. This approach is much more difficult because Φ_{12} is so complicated. However, since $M_{12} = M_{21}$, we do know the result of this calculation.

14.1 Check Your Understanding A current $I(t) = (5.0\ \text{A}) \sin((120\pi\ \text{rad/s})t)$ flows through the solenoid of part (b) of **Example 14.1**. What is the maximum emf induced in the surrounding coil?

14.2 | Self-Inductance and Inductors

Learning Objectives

By the end of this section, you will be able to:

- Correlate the rate of change of current to the induced emf created by that current in the same circuit
- Derive the self-inductance for a cylindrical solenoid
- Derive the self-inductance for a rectangular toroid

Mutual inductance arises when a current in one circuit produces a changing magnetic field that induces an emf in another circuit. But can the magnetic field affect the current in the original circuit that produced the field? The answer is yes, and this is the phenomenon called *self-inductance*.

Inductors

Figure 14.5 shows some of the magnetic field lines due to the current in a circular loop of wire. If the current is constant, the magnetic flux through the loop is also constant. However, if the current I were to vary with time—say, immediately after switch S is closed—then the magnetic flux Φ_m would correspondingly change. Then Faraday's law tells us that an emf ε would be induced in the circuit, where

$$\varepsilon = -\frac{d\Phi_m}{dt}. \tag{14.6}$$

Since the magnetic field due to a current-carrying wire is directly proportional to the current, the flux due to this field is also proportional to the current; that is,

$$\Phi_m \propto I. \tag{14.7}$$

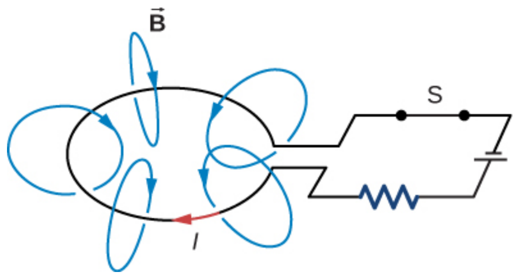


Figure 14.5 A magnetic field is produced by the current I in the loop. If I were to vary with time, the magnetic flux through the loop would also vary and an emf would be induced in the loop.

This can also be written as

$$\Phi_m = LI \tag{14.8}$$

where the constant of proportionality L is known as the **self-inductance** of the wire loop. If the loop has N turns, this equation becomes

$$N\Phi_m = LI. \tag{14.9}$$

By convention, the positive sense of the normal to the loop is related to the current by the right-hand rule, so in **Figure 14.5**, the normal points downward. With this convention, Φ_m is positive in **Equation 14.9**, so L *always has a positive value*.

For a loop with N turns, $\varepsilon = -Nd\Phi_m/dt,$ so the induced emf may be written in terms of the self-inductance as

$$\varepsilon = -L\frac{dI}{dt}. \tag{14.10}$$

When using this equation to determine L, it is easiest to ignore the signs of ε and $dI/dt,$ and calculate L as

$$L = \frac{|\varepsilon|}{|dI/dt|}.$$

Since self-inductance is associated with the magnetic field produced by a current, any configuration of conductors possesses self-inductance. For example, besides the wire loop, a long, straight wire has self-inductance, as does a coaxial cable. A coaxial cable is most commonly used by the cable television industry and may also be found connecting to your cable modem. Coaxial cables are used due to their ability to transmit electrical signals with minimal distortions. Coaxial cables have two long cylindrical conductors that possess current and a self-inductance that may have undesirable effects.

A circuit element used to provide self-inductance is known as an **inductor**. It is represented by the symbol shown in Figure 14.6, which resembles a coil of wire, the basic form of the inductor. Figure 14.7 shows several types of inductors commonly used in circuits.

Figure 14.6 Symbol used to represent an inductor in a circuit.

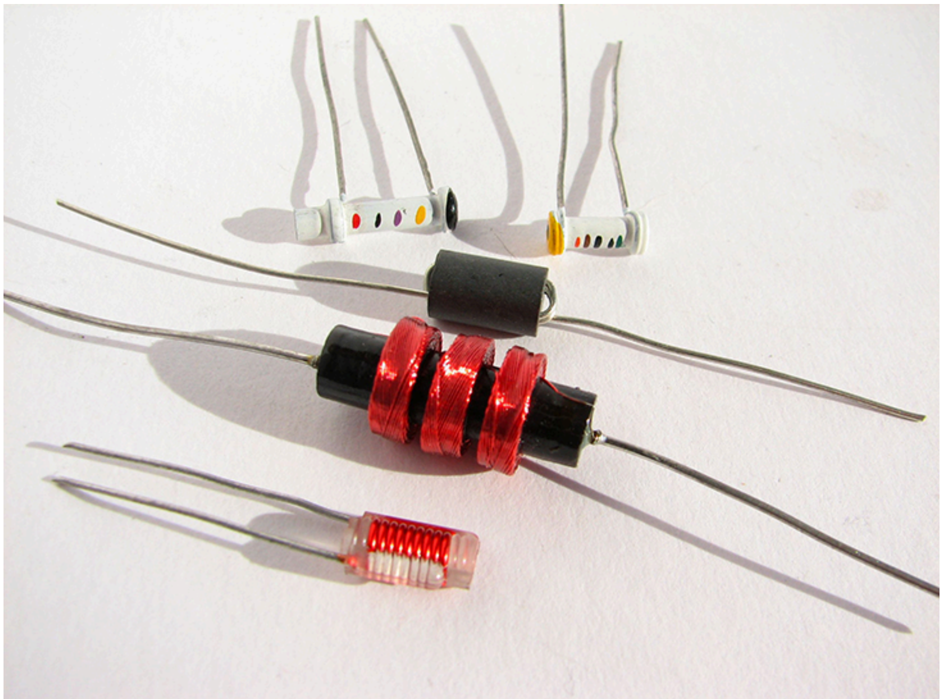


Figure 14.7 A variety of inductors. Whether they are encapsulated like the top three shown or wound around in a coil like the bottom-most one, each is simply a relatively long coil of wire. (credit: Windell Oskay)

In accordance with Lenz's law, the negative sign in Equation 14.10 indicates that the induced emf across an inductor always has a polarity that *opposes* the change in the current. For example, if the current flowing from A to B in Figure 14.8(a) were increasing, the induced emf (represented by the imaginary battery) would have the polarity shown in order to oppose the increase. If the current from A to B were decreasing, then the induced emf would have the opposite polarity, again to oppose the change in current (Figure 14.8(b)). Finally, if the current through the inductor were constant, no emf would be induced in the coil.

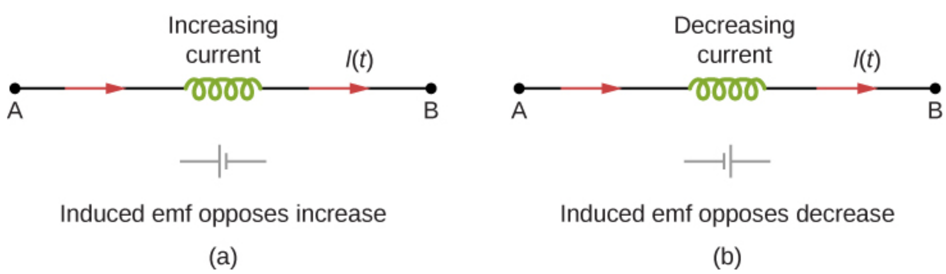


Figure 14.8 The induced emf across an inductor always acts to oppose the change in the current. This can be visualized as an imaginary battery causing current to flow to oppose the change in (a) and reinforce the change in (b).

One common application of inductance is to allow traffic signals to sense when vehicles are waiting at a street intersection. An electrical circuit with an inductor is placed in the road underneath the location where a waiting car will stop. The body of the car increases the inductance and the circuit changes, sending a signal to the traffic lights to change colors. Similarly, metal detectors used for airport security employ the same technique. A coil or inductor in the metal detector frame acts as both a transmitter and a receiver. The pulsed signal from the transmitter coil induces a signal in the receiver. The self-inductance of the circuit is affected by any metal object in the path (**Figure 14.9**). Metal detectors can be adjusted for sensitivity and can also sense the presence of metal on a person.

Figure 14.9 The familiar security gate at an airport not only detects metals, but can also indicate their approximate height above the floor. (credit: "Alexbuirds"/Wikimedia Commons)

Large induced voltages are found in camera flashes. Camera flashes use a battery, two inductors that function as a transformer, and a switching system or *oscillator* to induce large voltages. Recall from **Oscillations (http://cnx.org/content/m58360/latest/)** on oscillations that "oscillation" is defined as the fluctuation of a quantity, or repeated regular fluctuations of a quantity, between two extreme values around an average value. Also recall (from **Electromagnetic Induction** on electromagnetic induction) that we need a changing magnetic field, brought about by a changing current, to induce a voltage in another coil. The oscillator system does this many times as the battery voltage is boosted to over 1000 volts. (You may hear the high-pitched whine from the transformer as the capacitor is being charged.) A capacitor stores the high voltage for later use in powering the flash.

Example 14.2

Self-Inductance of a Coil

An induced emf of 2.0 V is measured across a coil of 50 closely wound turns while the current through it increases uniformly from 0.0 to 5.0 A in 0.10 s. (a) What is the self-inductance of the coil? (b) With the current at 5.0 A, what is the flux through each turn of the coil?

Strategy

Both parts of this problem give all the information needed to solve for the self-inductance in part (a) or the flux through each turn of the coil in part (b). The equations needed are **Equation 14.10** for part (a) and **Equation 14.9** for part (b).

Solution

a. Ignoring the negative sign and using magnitudes, we have, from **Equation 14.10**,

$$L = \frac{\varepsilon}{dI/dt} = \frac{2.0\text{ V}}{5.0\text{ A}/0.10\text{ s}} = 4.0 \times 10^{-2}\text{ H}.$$

b. From **Equation 14.9**, the flux is given in terms of the current by $\Phi_m = LI/N$, so

$$\Phi_m = \frac{(4.0 \times 10^{-2}\text{ H})(5.0\text{ A})}{50\text{ turns}} = 4.0 \times 10^{-3}\text{ Wb}.$$

Significance

The self-inductance and flux calculated in parts (a) and (b) are typical values for coils found in contemporary devices. If the current is not changing over time, the flux is not changing in time, so no emf is induced.

14.2 Check Your Understanding Current flows through the inductor in **Figure 14.8** from B to A instead of from A to B as shown. Is the current increasing or decreasing in order to produce the emf given in diagram (a)? In diagram (b)?

14.3 Check Your Understanding A changing current induces an emf of 10 V across a 0.25-H inductor. What is the rate at which the current is changing?

A good approach for calculating the self-inductance of an inductor consists of the following steps:

Problem-Solving Strategy: Self-Inductance

1. Assume a current I is flowing through the inductor.
2. Determine the magnetic field $\vec{\mathbf{B}}$ produced by the current. If there is appropriate symmetry, you may be able to do this with Ampère's law.
3. Obtain the magnetic flux, Φ_{m}.
4. With the flux known, the self-inductance can be found from **Equation 14.9**, $L = N\Phi_{\mathrm{m}}/I$.

To demonstrate this procedure, we now calculate the self-inductances of two inductors.

Cylindrical Solenoid

Consider a long, cylindrical solenoid with length l, cross-sectional area A, and N turns of wire. We assume that the length of the solenoid is so much larger than its diameter that we can take the magnetic field to be $B = \mu_0 nI$ throughout the interior of the solenoid, that is, we ignore end effects in the solenoid. With a current I flowing through the coils, the magnetic field produced within the solenoid is

$$B = \mu_0 \left(\frac{N}{l}\right) I, \tag{14.11}$$

so the magnetic flux through one turn is

$$\Phi_{\mathrm{m}} = BA = \frac{\mu_0 NA}{l} I. \tag{14.12}$$

Using **Equation 14.9**, we find for the self-inductance of the solenoid,

$$L_{\text{solenoid}} = \frac{N\Phi_{\mathrm{m}}}{I} = \frac{\mu_0 N^2 A}{l}. \tag{14.13}$$

If $n = N/l$ is the number of turns per unit length of the solenoid, we may write **Equation 14.13** as

$$L = \mu_0 \left(\frac{N}{l}\right)^2 Al = \mu_0 n^2 Al = \mu_0 n^2 (V), \tag{14.14}$$

where $V = Al$ is the volume of the solenoid. Notice that *the self-inductance of a long solenoid depends only on its physical properties* (such as the number of turns of wire per unit length and the volume), and not on the magnetic field or the current.

This is true for inductors in general.

Rectangular Toroid

A toroid with a rectangular cross-section is shown in **Figure 14.10**. The inner and outer radii of the toroid are R_1 and R_2, and h is the height of the toroid. Applying Ampère's law in the same manner as we did in **Example 13.8** for a toroid with a circular cross-section, we find the magnetic field inside a rectangular toroid is also given by

$$B = \frac{\mu_0 NI}{2\pi r}, \quad (14.15)$$

where r is the distance from the central axis of the toroid. Because the field changes within the toroid, we must calculate the flux by integrating over the toroid's cross-section. Using the infinitesimal cross-sectional area element $da = h\,dr$ shown in **Figure 14.10**, we obtain

$$\Phi_{\text{m}} = \int B\,da = \int_{R_1}^{R_2} \left(\frac{\mu_0 NI}{2\pi r}\right)(hdr) = \frac{\mu_0 NhI}{2\pi} \ln \frac{R_2}{R_1}. \quad (14.16)$$

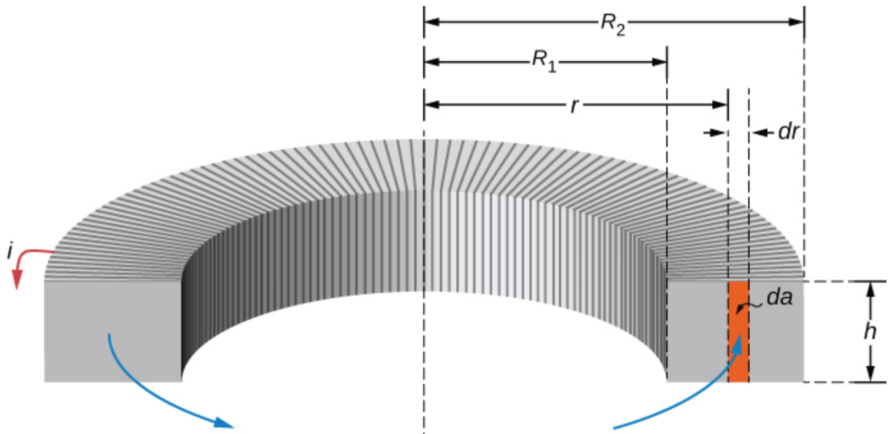


Figure 14.10 Calculating the self-inductance of a rectangular toroid.

Now from **Equation 14.16**, we obtain for the self-inductance of a rectangular toroid

$$L = \frac{N\Phi_{\text{m}}}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1}. \quad (14.17)$$

As expected, the self-inductance is a constant determined by only the physical properties of the toroid.

14.4 Check Your Understanding (a) Calculate the self-inductance of a solenoid that is tightly wound with wire of diameter 0.10 cm, has a cross-sectional area of $0.90\,\text{cm}^2$, and is 40 cm long. (b) If the current through the solenoid decreases uniformly from 10 to 0 A in 0.10 s, what is the emf induced between the ends of the solenoid?

14.5 Check Your Understanding (a) What is the magnetic flux through one turn of a solenoid of self-inductance $8.0 \times 10^{-5}\,\text{H}$ when a current of 3.0 A flows through it? Assume that the solenoid has 1000 turns and is wound from wire of diameter 1.0 mm. (b) What is the cross-sectional area of the solenoid?

14.3 | Energy in a Magnetic Field

Learning Objectives

By the end of this section, you will be able to:

- Explain how energy can be stored in a magnetic field
- Derive the equation for energy stored in a coaxial cable given the magnetic energy density

The energy of a capacitor is stored in the electric field between its plates. Similarly, an inductor has the capability to store energy, but in its magnetic field. This energy can be found by integrating the **magnetic energy density**,

$$u_{\text{m}} = \frac{B^2}{2\mu_0} \tag{14.18}$$

over the appropriate volume. To understand where this formula comes from, let's consider the long, cylindrical solenoid of the previous section. Again using the infinite solenoid approximation, we can assume that the magnetic field is essentially constant and given by $B = \mu_0 nI$ everywhere inside the solenoid. Thus, the energy stored in a solenoid or the magnetic energy density times volume is equivalent to

$$U = u_{\text{m}}(V) = \frac{(\mu_0 nI)^2}{2\mu_0}(Al) = \frac{1}{2}\left(\mu_0 n^2 Al\right)I^2. \tag{14.19}$$

With the substitution of **Equation 14.14**, this becomes

$$U = \frac{1}{2}LI^2. \tag{14.20}$$

Although derived for a special case, this equation gives the energy stored in the magnetic field of *any* inductor. We can see this by considering an arbitrary inductor through which a changing current is passing. At any instant, the magnitude of the induced emf is $\varepsilon = Ldi/dt,$ so the power absorbed by the inductor is

$$P = \varepsilon i = L\frac{di}{dt}i. \tag{14.21}$$

The total energy stored in the magnetic field when the current increases from 0 to I in a time interval from 0 to t can be determined by integrating this expression:

$$U = \int_0^t P dt' = \int_0^t L\frac{di}{dt'}i dt' = L\int_0^l i di = \frac{1}{2}LI^2. \tag{14.22}$$

Example 14.3

Self-Inductance of a Coaxial Cable

Figure 14.11 shows two long, concentric cylindrical shells of radii R_1 and R_2. As discussed in **Capacitance** on capacitance, this configuration is a simplified representation of a coaxial cable. The capacitance per unit length of the cable has already been calculated. Now (a) determine the magnetic energy stored per unit length of the coaxial cable and (b) use this result to find the self-inductance per unit length of the cable.

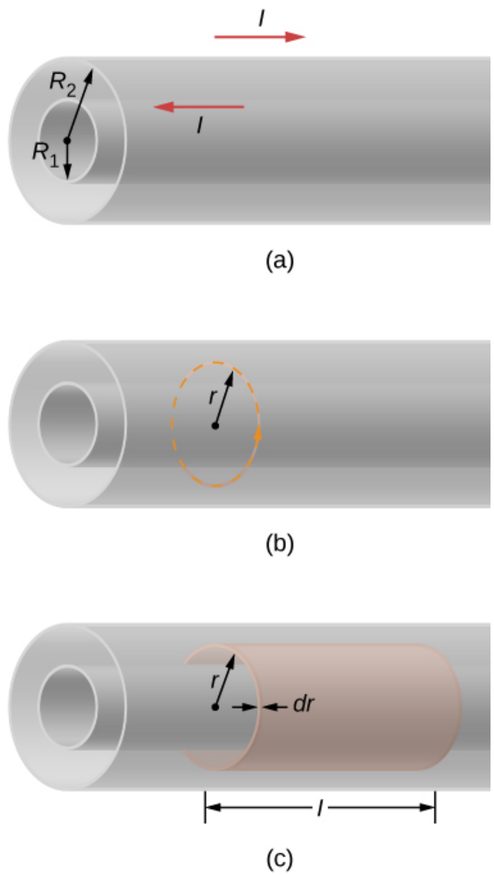


Figure 14.11 (a) A coaxial cable is represented here by two hollow, concentric cylindrical conductors along which electric current flows in opposite directions. (b) The magnetic field between the conductors can be found by applying Ampère's law to the dashed path. (c) The cylindrical shell is used to find the magnetic energy stored in a length *l* of the cable.

Strategy

The magnetic field both inside and outside the coaxial cable is determined by Ampère's law. Based on this magnetic field, we can use **Equation 14.22** to calculate the energy density of the magnetic field. The magnetic energy is calculated by an integral of the magnetic energy density times the differential volume over the cylindrical shell. After the integration is carried out, we have a closed-form solution for part (a). The self-inductance per unit length is determined based on this result and **Equation 14.22**.

Solution

a. We determine the magnetic field between the conductors by applying Ampère's law to the dashed circular path shown in **Figure 14.11**(b). Because of the cylindrical symmetry, $\vec{\mathbf{B}}$ is constant along the path, and

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r) = \mu_0 I.$$

This gives us

$$B = \frac{\mu_0 I}{2\pi r}.$$

In the region outside the cable, a similar application of Ampère's law shows that $B = 0$, since no net current crosses the area bounded by a circular path where $r > R_2$. This argument also holds when $r < R_1$; that is, in the region within the inner cylinder. All the magnetic energy of the cable is therefore stored between the two conductors. Since the energy density of the magnetic field is

$$u_{\text{m}} = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2},$$

the energy stored in a cylindrical shell of inner radius *r*, outer radius $r + dr$, and length *l* (see part (c) of the figure) is

$$u_{\text{m}} = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2}.$$

Thus, the total energy of the magnetic field in a length *l* of the cable is

$$U = \int_{R_1}^{R_2} dU = \int_{R_1}^{R_2} \frac{\mu_0 I^2}{8\pi^2 r^2}(2\pi r l)dr = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{R_2}{R_1},$$

and the energy per unit length is $(\mu_0 I^2/4\pi)\ln(R_2/R_1)$.

b. From **Equation 14.22**,

$$U = \frac{1}{2}LI^2,$$

where *L* is the self-inductance of a length *l* of the coaxial cable. Equating the previous two equations, we find that the self-inductance per unit length of the cable is

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{R_2}{R_1}.$$

Significance

The inductance per unit length depends only on the inner and outer radii as seen in the result. To increase the inductance, we could either increase the outer radius (R_2) or decrease the inner radius (R_1). In the limit as the two radii become equal, the inductance goes to zero. In this limit, there is no coaxial cable. Also, the magnetic energy per unit length from part (a) is proportional to the square of the current.

14.6 Check Your Understanding How much energy is stored in the inductor of **Example 14.2** after the current reaches its maximum value?

14.4 | RL Circuits

Learning Objectives

By the end of this section, you will be able to:

- Analyze circuits that have an inductor and resistor in series
- Describe how current and voltage exponentially grow or decay based on the initial conditions

A circuit with resistance and self-inductance is known as an *RL* circuit. **Figure 14.12**(a) shows an *RL* circuit consisting of a resistor, an inductor, a constant source of emf, and switches S_1 and S_2. When S_1 is closed, the circuit is equivalent to a single-loop circuit consisting of a resistor and an inductor connected across a source of emf (**Figure 14.12**(b)). When

S_1 is opened and S_2 is closed, the circuit becomes a single-loop circuit with only a resistor and an inductor (**Figure 14.12**(c)).

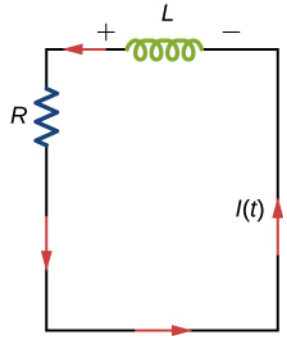


Figure 14.12 (a) An *RL* circuit with switches S_1 and S_2. (b) The equivalent circuit with S_1 closed and S_2 open. (c) The equivalent circuit after S_1 is opened and S_2 is closed.

We first consider the *RL* circuit of **Figure 14.12**(b). Once S_1 is closed and S_2 is open, the source of emf produces a current in the circuit. If there were no self-inductance in the circuit, the current would rise immediately to a steady value of ε/R. However, from Faraday's law, the increasing current produces an emf $V_L = -L(dI/dt)$ across the inductor. In accordance with Lenz's law, the induced emf counteracts the increase in the current and is directed as shown in the figure. As a result, *I(t)* starts at zero and increases asymptotically to its final value.

Applying Kirchhoff's loop rule to this circuit, we obtain

$$\varepsilon - L\frac{dI}{dt} - IR = 0, \tag{14.23}$$

which is a first-order differential equation for *I(t)*. Notice its similarity to the equation for a capacitor and resistor in series (See ***RC* Circuits**). Similarly, the solution to **Equation 14.23** can be found by making substitutions in the equations relating the capacitor to the inductor. This gives

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau_L}\right), \tag{14.24}$$

where

$$\tau_L = L/R \tag{14.25}$$

is the **inductive time constant** of the circuit.

The current *I(t)* is plotted in **Figure 14.13**(a). It starts at zero, and as $t \to \infty$, *I(t)* approaches ε/R asymptotically. The induced emf $V_L(t)$ is directly proportional to *dI/dt*, or the slope of the curve. Hence, while at its greatest immediately after the switches are thrown, the induced emf decreases to zero with time as the current approaches its final value of ε/R. The circuit then becomes equivalent to a resistor connected across a source of emf.

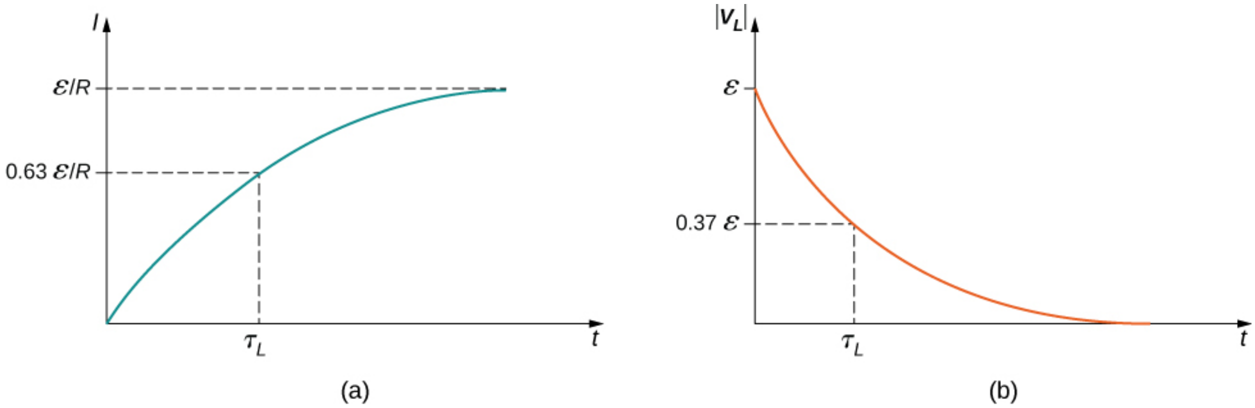


Figure 14.13 Time variation of (a) the electric current and (b) the magnitude of the induced voltage across the coil in the circuit of **Figure 14.12**(b).

The energy stored in the magnetic field of an inductor is

$$U_L = \frac{1}{2}LI^2. \tag{14.26}$$

Thus, as the current approaches the maximum current ε/R, the stored energy in the inductor increases from zero and asymptotically approaches a maximum of $L(\varepsilon/R)^2/2$.

The time constant τ_L tells us how rapidly the current increases to its final value. At $t = \tau_L,$ the current in the circuit is, from **Equation 14.24**,

$$I(\tau_L) = \frac{\varepsilon}{R}(1 - e^{-1}) = 0.63\frac{\varepsilon}{R}, \tag{14.27}$$

which is 63% of the final value ε/R. The smaller the inductive time constant $\tau_L = L/R,$ the more rapidly the current approaches ε/R.

We can find the time dependence of the induced voltage across the inductor in this circuit by using $V_L(t) = -L(dI/dt)$ and **Equation 14.24**:

$$V_L(t) = -L\frac{dI}{dt} = -\varepsilon e^{-t/\tau_L}. \tag{14.28}$$

The magnitude of this function is plotted in **Figure 14.13**(b). The greatest value of $L(dI/dt)$ is $\varepsilon;$ it occurs when *dI/dt* is greatest, which is immediately after S_1 is closed and S_2 is opened. In the approach to steady state, *dI/dt* decreases to zero. As a result, the voltage across the inductor also vanishes as $t \to \infty$.

The time constant τ_L also tells us how quickly the induced voltage decays. At $t = \tau_L,$ the magnitude of the induced voltage is

$$|V_L(\tau_L)| = \varepsilon e^{-1} = 0.37\varepsilon = 0.37V(0). \tag{14.29}$$

The voltage across the inductor therefore drops to about 37% of its initial value after one time constant. The shorter the time constant $\tau_L,$ the more rapidly the voltage decreases.

After enough time has elapsed so that the current has essentially reached its final value, the positions of the switches in **Figure 14.12**(a) are reversed, giving us the circuit in part (c). At $t = 0,$ the current in the circuit is $I(0) = \varepsilon/R.$ With Kirchhoff's loop rule, we obtain

$$IR + L\frac{dI}{dt} = 0. \tag{14.30}$$

The solution to this equation is similar to the solution of the equation for a discharging capacitor, with similar substitutions. The current at time *t* is then

$$I(t) = \frac{\varepsilon}{R} e^{-t/\tau_L}. \qquad (14.31)$$

The current starts at $I(0) = \varepsilon/R$ and decreases with time as the energy stored in the inductor is depleted (**Figure 14.14**).

The time dependence of the voltage across the inductor can be determined from $V_L = -L(dI/dt)$:

$$V_L(t) = \varepsilon e^{-t/\tau L}. \qquad (14.32)$$

This voltage is initially $V_L(0) = \varepsilon$, and it decays to zero like the current. The energy stored in the magnetic field of the inductor, $LI^2/2$, also decreases exponentially with time, as it is dissipated by Joule heating in the resistance of the circuit.

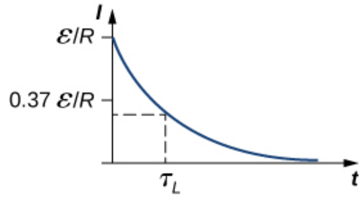


Figure 14.14 Time variation of electric current in the *RL* circuit of **Figure 14.12**(c). The induced voltage across the coil also decays exponentially.

Example 14.4

An *RL* Circuit with a Source of emf

In the circuit of **Figure 14.12**(a), let $\varepsilon = 2.0V$, $R = 4.0\,\Omega$, and $L = 4.0\,\text{H}$. With S_1 closed and S_2 open (**Figure 14.12**(b)), (a) what is the time constant of the circuit? (b) What are the current in the circuit and the magnitude of the induced emf across the inductor at $t = 0$, at $t = 2.0\tau_L$, and as $t \to \infty$?

Strategy

The time constant for an inductor and resistor in a series circuit is calculated using **Equation 14.25**. The current through and voltage across the inductor are calculated by the scenarios detailed from **Equation 14.24** and **Equation 14.32**.

Solution

a. The inductive time constant is

$$\tau_L = \frac{L}{R} = \frac{4.0\,\text{H}}{4.0\,\Omega} = 1.0\,\text{s}.$$

b. The current in the circuit of **Figure 14.12**(b) increases according to **Equation 14.24**:

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau_L}).$$

At $t = 0$,

$$(1 - e^{-t/\tau_L}) = (1 - 1) = 0; \text{ so } I(0) = 0.$$

At $t = 2.0\tau_L$ and $t \to \infty$, we have, respectively,

$$I(2.0\tau_L) = \frac{\varepsilon}{R}(1 - e^{-2.0}) = (0.50\,\text{A})(0.86) = 0.43\,\text{A},$$

and

$$I(\infty) = \frac{\varepsilon}{R} = 0.50\,\text{A}.$$

From **Equation 14.32**, the magnitude of the induced emf decays as

$$|V_L(t)| = \varepsilon e^{-t/\tau_L}.$$

At $t = 0,\ t = 2.0\tau_L,$ and as $t \to \infty,$ we obtain

$$\begin{aligned} |V_L(0)| &= \varepsilon = 2.0\text{ V}, \\ |V_L(2.0\tau_L)| &= (2.0\text{ V})\,e^{-2.0} = 0.27\text{ V} \\ &\text{and} \\ |V_L(\infty)| &= 0. \end{aligned}$$

Significance

If the time of the measurement were much larger than the time constant, we would not see the decay or growth of the voltage across the inductor or resistor. The circuit would quickly reach the asymptotic values for both of these. See **Figure 14.15**.

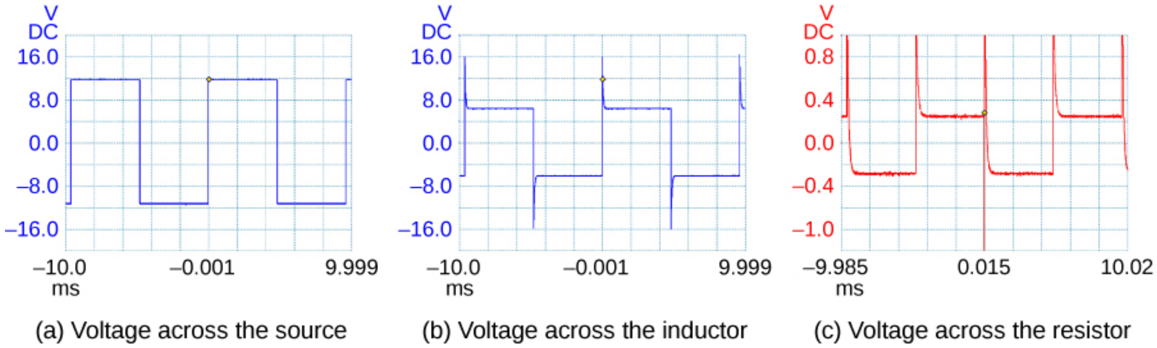


Figure 14.15 A generator in an *RL* circuit produces a square-pulse output in which the voltage oscillates between zero and some set value. These oscilloscope traces show (a) the voltage across the source; (b) the voltage across the inductor; (c) the voltage across the resistor.

Example 14.5

An *RL* Circuit without a Source of emf

After the current in the *RL* circuit of **Example 14.4** has reached its final value, the positions of the switches are reversed so that the circuit becomes the one shown in **Figure 14.12**(c). (a) How long does it take the current to drop to half its initial value? (b) How long does it take before the energy stored in the inductor is reduced to 1.0% of its maximum value?

Strategy

The current in the inductor will now decrease as the resistor dissipates this energy. Therefore, the current falls as an exponential decay. We can also use that same relationship as a substitution for the energy in an inductor formula to find how the energy decreases at different time intervals.

Solution

a. With the switches reversed, the current decreases according to

$$I(t) = \frac{\varepsilon}{R}e^{-t/\tau_L} = I(0)e^{-t/\tau_L}.$$

At a time t when the current is one-half its initial value, we have

$$I(t) = 0.50I(0) \text{ so } e^{-t/\tau_L} = 0.50,$$

and

$$t = -[\ln(0.50)]\tau_L = 0.69(1.0\text{ s}) = 0.69\text{ s},$$

where we have used the inductive time constant found in **Example 14.4**.

b. The energy stored in the inductor is given by

$$U_L(t) = \frac{1}{2}L[I(t)]^2 = \frac{1}{2}L\left(\frac{\varepsilon}{R}e^{-t/\tau_L}\right)^2 = \frac{L\varepsilon^2}{2R^2}e^{-2t/\tau_L}.$$

If the energy drops to 1.0% of its initial value at a time t, we have

$$U_L(t) = (0.010)U_L(0) \text{ or } \frac{L\varepsilon^2}{2R^2}e^{-2t/\tau_L} = (0.010)\frac{L\varepsilon^2}{2R^2}.$$

Upon canceling terms and taking the natural logarithm of both sides, we obtain

$$-\frac{2t}{\tau_L} = \ln(0.010),$$

so

$$t = -\frac{1}{2}\tau_L \ln(0.010).$$

Since $\tau_L = 1.0\text{ s}$, the time it takes for the energy stored in the inductor to decrease to 1.0% of its initial value is

$$t = -\frac{1}{2}(1.0\text{ s})\ln(0.010) = 2.3\text{ s}.$$

Significance

This calculation only works if the circuit is at maximum current in situation (b) prior to this new situation. Otherwise, we start with a lower initial current, which will decay by the same relationship.

14.7 Check Your Understanding Verify that *RC* and *L/R* have the dimensions of time.

14.8 Check Your Understanding (a) If the current in the circuit of in **Figure 14.12**(b) increases to 90% of its final value after 5.0 s, what is the inductive time constant? (b) If $R = 20\,\Omega$, what is the value of the self-inductance? (c) If the $20\text{-}\Omega$ resistor is replaced with a $100\text{-}\Omega$ resister, what is the time taken for the current to reach 90% of its final value?

14.9 Check Your Understanding For the circuit of in **Figure 14.12**(b), show that when steady state is reached, the difference in the total energies produced by the battery and dissipated in the resistor is equal to the energy stored in the magnetic field of the coil.

14.5 | Oscillations in an LC Circuit

Learning Objectives

By the end of this section, you will be able to:

- Explain why charge or current oscillates between a capacitor and inductor, respectively, when wired in series
- Describe the relationship between the charge and current oscillating between a capacitor and inductor wired in series

It is worth noting that both capacitors and inductors store energy, in their electric and magnetic fields, respectively. A circuit containing both an inductor (*L*) and a capacitor (*C*) can oscillate without a source of emf by shifting the energy stored in the circuit between the electric and magnetic fields. Thus, the concepts we develop in this section are directly applicable to the exchange of energy between the electric and magnetic fields in electromagnetic waves, or light. We start with an idealized circuit of zero resistance that contains an inductor and a capacitor, an ***LC* circuit**.

An *LC* circuit is shown in **Figure 14.16**. If the capacitor contains a charge q_0 before the switch is closed, then all the energy of the circuit is initially stored in the electric field of the capacitor (**Figure 14.16**(a)). This energy is

$$U_C = \frac{1}{2}\frac{q_0^2}{C}. \tag{14.33}$$

When the switch is closed, the capacitor begins to discharge, producing a current in the circuit. The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.

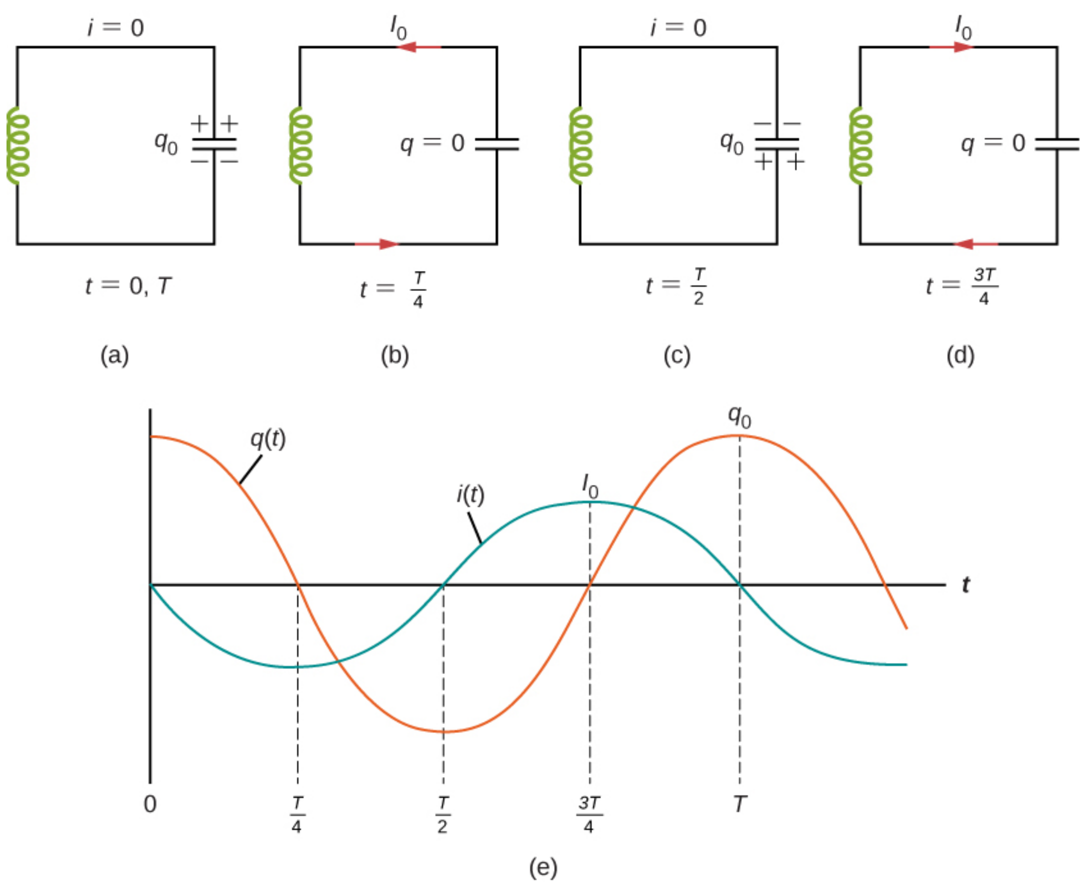


Figure 14.16 (a–d) The oscillation of charge storage with changing directions of current in an *LC* circuit. (e) The graphs show the distribution of charge and current between the capacitor and inductor.

In Figure 14.16(b), the capacitor is completely discharged and all the energy is stored in the magnetic field of the inductor. At this instant, the current is at its maximum value I_0 and the energy in the inductor is

$$U_L = \frac{1}{2}LI_0^2. \tag{14.34}$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor:

$$\frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}LI_0^2. \tag{14.35}$$

At an arbitrary time when the capacitor charge is *q(t)* and the current is *i(t)*, the total energy *U* in the circuit is given by

$$\frac{q^2(t)}{2C} + \frac{Li^2(t)}{2}.$$

Because there is no energy dissipation,

$$U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2 = \frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}LI_0^2. \tag{14.36}$$

After reaching its maximum I_0, the current *i(t)* continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes, and the overall effect is a transfer of energy from the inductor *back* to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_0. However, as Figure 14.16(c) shows, the capacitor plates are charged *opposite* to what they were initially.

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged, as shown in Figure 14.16(d). Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored.

We have followed the circuit through one complete cycle. Its electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring. In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $mv^2/2$, and the spring, which has potential energy $kx^2/2$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an *LC* circuit with no resistance would continue forever if undisturbed; however, this ideal zero-resistance *LC* circuit is not practical, and any *LC* circuit will have at least a small resistance, which will radiate and lose energy over time.

The frequency of the oscillations in a resistance-free *LC* circuit may be found by analogy with the mass-spring system. For the circuit, $i(t) = dq(t)/dt$, the total electromagnetic energy *U* is

$$U = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}. \tag{14.37}$$

For the mass-spring system, $v(t) = dx(t)/dt$, the total mechanical energy *E* is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{14.38}$$

The equivalence of the two systems is clear. To go from the mechanical to the electromagnetic system, we simply replace *m* by *L*, *v* by *i*, *k* by 1/*C*, and *x* by *q*. Now *x(t)* is given by

$$x(t) = A\cos(\omega t + \phi) \tag{14.39}$$

where $\omega = \sqrt{k/m}$. Hence, the charge on the capacitor in an *LC* circuit is given by

$$q(t) = q_0\cos(\omega t + \phi) \tag{14.40}$$

where the angular frequency of the oscillations in the circuit is

$$\omega = \sqrt{\frac{1}{LC}}. \tag{14.41}$$

Finally, the current in the *LC* circuit is found by taking the time derivative of *q(t)*:

$$i(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin(\omega t + \phi). \tag{14.42}$$

The time variations of q and I are shown in **Figure 14.16**(e) for $\phi = 0$.

Example 14.6

An *LC* Circuit

In an *LC* circuit, the self-inductance is 2.0×10^{-2} H and the capacitance is 8.0×10^{-6} F. At $t = 0$, all of the energy is stored in the capacitor, which has charge 1.2×10^{-5} C. (a) What is the angular frequency of the oscillations in the circuit? (b) What is the maximum current flowing through circuit? (c) How long does it take the capacitor to become completely discharged? (d) Find an equation that represents *q(t)*.

Strategy

The angular frequency of the *LC* circuit is given by **Equation 14.41**. To find the maximum current, the maximum energy in the capacitor is set equal to the maximum energy in the inductor. The time for the capacitor to become discharged if it is initially charged is a quarter of the period of the cycle, so if we calculate the period of the oscillation, we can find out what a quarter of that is to find this time. Lastly, knowing the initial charge and angular frequency, we can set up a cosine equation to find $q(t)$.

Solution

a. From **Equation 14.41**, the angular frequency of the oscillations is

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(2.0 \times 10^{-2}\ \text{H})(8.0 \times 10^{-6}\ \text{F})}} = 2.5 \times 10^{3}\ \text{rad/s}.$$

b. The current is at its maximum I_0 when all the energy is stored in the inductor. From the law of energy conservation,

$$\frac{1}{2} L I_0^2 = \frac{1}{2} \frac{q_0^2}{C},$$

so

$$I_0 = \sqrt{\frac{1}{LC}} q_0 = (2.5 \times 10^{3}\ \text{rad/s})(1.2 \times 10^{-5}\ \text{C}) = 3.0 \times 10^{-2}\ \text{A}.$$

This result can also be found by an analogy to simple harmonic motion, where current and charge are the velocity and position of an oscillator.

c. The capacitor becomes completely discharged in one-fourth of a cycle, or during a time $T/4$, where T is the period of the oscillations. Since

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.5 \times 10^{3}\ \text{rad/s}} = 2.5 \times 10^{-3}\ \text{s},$$

the time taken for the capacitor to become fully discharged is $(2.5 \times 10^{-3}\ \text{s})/4 = 6.3 \times 10^{-4}\ \text{s}$.

d. The capacitor is completely charged at $t = 0$, so $q(0) = q_0$. Using **Equation 14.20**, we obtain

$$q(0) = q_0 = q_0 \cos \phi.$$

Thus, $\phi = 0$, and

$$q(t) = (1.2 \times 10^{-5}\ \text{C})\cos(2.5 \times 10^{3}\, t).$$

Significance

The energy relationship set up in part (b) is not the only way we can equate energies. At most times, some energy is stored in the capacitor and some energy is stored in the inductor. We can put both terms on each side of the equation. By examining the circuit only when there is no charge on the capacitor or no current in the inductor, we simplify the energy equation.

14.10 Check Your Understanding The angular frequency of the oscillations in an *LC* circuit is 2.0×10^3 rad/s. (a) If $L = 0.10\ \text{H}$, what is *C*? (b) Suppose that at $t = 0$, all the energy is stored in the inductor. What is the value of ϕ? (c) A second identical capacitor is connected in parallel with the original capacitor. What is the angular frequency of this circuit?

14.6 | RLC Series Circuits

Learning Objectives

By the end of this section, you will be able to:

- Determine the angular frequency of oscillation for a resistor, inductor, capacitor (RLC) series circuit
- Relate the RLC circuit to a damped spring oscillation

When the switch is closed in the ***RLC* circuit** of **Figure 14.17**(a), the capacitor begins to discharge and electromagnetic energy is dissipated by the resistor at a rate $i^2 R$. With *U* given by **Equation 14.19**, we have

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2 R \qquad (14.43)$$

where *i* and *q* are time-dependent functions. This reduces to

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0. \qquad (14.44)$$

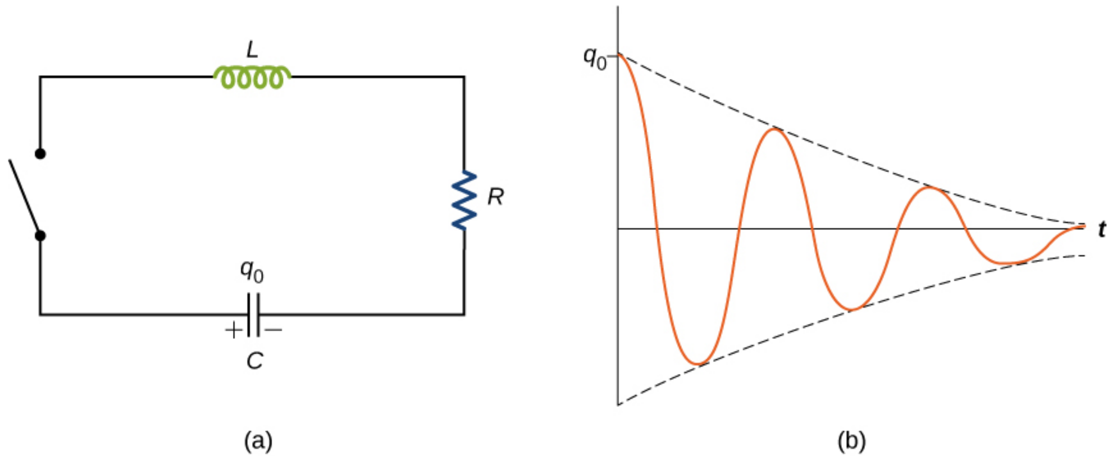


Figure 14.17 (a) An *RLC* circuit. Electromagnetic oscillations begin when the switch is closed. The capacitor is fully charged initially. (b) Damped oscillations of the capacitor charge are shown in this curve of charge versus time, or *q* versus *t*. The capacitor contains a charge q_0 before the switch is closed.

This equation is analogous to

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

which is the equation of motion for a *damped mass-spring system* (you first encountered this equation in **Oscillations (http://cnx.org/content/m58360/latest/)**). As we saw in that chapter, it can be shown that the solution to this differential equation takes three forms, depending on whether the angular frequency of the undamped spring is greater than, equal to, or less than $b/2m$. Therefore, the result can be underdamped $(\sqrt{k/m} > b/2m)$, critically damped $(\sqrt{k/m} = b/2m)$, or overdamped $(\sqrt{k/m} < b/2m)$. By analogy, the solution $q(t)$ to the *RLC* differential equation has the same feature. Here we look only at the case of under-damping. By replacing m by L, b by R, k by $1/C$, and x by q in **Equation 14.44**, and assuming $\sqrt{1/LC} > R/2L$, we obtain

$$q(t) = q_0 e^{-Rt/2L} \cos(\omega' t + \phi) \tag{14.45}$$

where the angular frequency of the oscillations is given by

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{14.46}$$

This underdamped solution is shown in **Figure 14.17**(b). Notice that the amplitude of the oscillations decreases as energy is dissipated in the resistor. **Equation 14.45** can be confirmed experimentally by measuring the voltage across the capacitor as a function of time. This voltage, multiplied by the capacitance of the capacitor, then gives $q(t)$.

Try an **interactive circuit construction kit (https://openstaxcollege.org/l/21phetcirconstr)** that allows you to graph current and voltage as a function of time. You can add inductors and capacitors to work with any combination of *R*, *L*, and *C* circuits with both dc and ac sources.

Try out a **circuit-based java applet website (https://openstaxcollege.org/l/21cirphysbascur)** that has many problems with both dc and ac sources that will help you practice circuit problems.

14.11 Check Your Understanding In an *RLC* circuit, $L = 5.0\,\text{mH}$, $C = 6.0\mu\text{F}$, and $R = 200\,\Omega$. (a) Is the circuit underdamped, critically damped, or overdamped? (b) If the circuit starts oscillating with a charge of $3.0 \times 10^{-3}\ \text{C}$ on the capacitor, how much energy has been dissipated in the resistor by the time the oscillations cease?

CHAPTER 14 REVIEW

KEY TERMS

henry (H) unit of inductance, $1 \text{ H} = 1 \, \Omega \cdot \text{s}$; it is also expressed as a volt second per ampere

inductance property of a device that tells how effectively it induces an emf in another device

inductive time constant denoted by τ , the characteristic time given by quantity *L*/*R* of a particular series *RL* circuit

inductor part of an electrical circuit to provide self-inductance, which is symbolized by a coil of wire

***LC* circuit** circuit composed of an ac source, inductor, and capacitor

magnetic energy density energy stored per volume in a magnetic field

mutual inductance geometric quantity that expresses how effective two devices are at inducing emfs in one another

***RLC* circuit** circuit with an ac source, resistor, inductor, and capacitor all in series.

self-inductance effect of the device inducing emf in itself

KEY EQUATIONS

Mutual inductance by flux	$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}$
Mutual inductance in circuits	$\varepsilon_1 = -M \frac{dI_2}{dt}$
Self-inductance in terms of magnetic flux	$N\Phi_\text{m} = LI$
Self-inductance in terms of emf	$\varepsilon = -L \frac{dI}{dt}$
Self-inductance of a solenoid	$L_\text{solenoid} = \frac{\mu_0 N^2 A}{l}$
Self-inductance of a toroid	$L_\text{toroid} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1}.$
Energy stored in an inductor	$U = \frac{1}{2} L I^2$
Current as a function of time for a *RL* circuit	$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau_L})$
Time constant for a *RL* circuit	$\tau_L = L/R$
Charge oscillation in *LC* circuits	$q(t) = q_0 \cos(\omega t + \phi)$
Angular frequency in *LC* circuits	$\omega = \sqrt{\frac{1}{LC}}$
Current oscillations in *LC* circuits	$i(t) = -\omega q_0 \sin(\omega t + \phi)$
Charge as a function of time in *RLC* circuit	$q(t) = q_0 e^{-Rt/2L} \cos(\omega' t + \phi)$
Angular frequency in *RLC* circuit	$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

SUMMARY

14.1 Mutual Inductance

- Inductance is the property of a device that expresses how effectively it induces an emf in another device.
- Mutual inductance is the effect of two devices inducing emfs in each other.
- A change in current dI_1/dt in one circuit induces an emf (ε_2) in the second:

$$\varepsilon_2 = -M\frac{dI1}{dt},$$

  where M is defined to be the mutual inductance between the two circuits and the minus sign is due to Lenz's law.
- Symmetrically, a change in current dI_2/dt through the second circuit induces an emf (ε_1) in the first:

$$\varepsilon_1 = -M\frac{dI_2}{dt},$$

  where M is the same mutual inductance as in the reverse process.

14.2 Self-Inductance and Inductors

- Current changes in a device induce an emf in the device itself, called self-inductance,

$$\varepsilon = -L\frac{dI}{dt},$$

  where L is the self-inductance of the inductor and dI/dt is the rate of change of current through it. The minus sign indicates that emf opposes the change in current, as required by Lenz's law. The unit of self-inductance and inductance is the henry (H), where $1 \text{ H} = 1\,\Omega \cdot \text{s}$.
- The self-inductance of a solenoid is

$$L = \frac{\mu_0 N^2 A}{l},$$

  where N is its number of turns in the solenoid, A is its cross-sectional area, l is its length, and $\mu_0 = 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ is the permeability of free space.
- The self-inductance of a toroid is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\frac{R_2}{R_1},$$

  where N is its number of turns in the toroid, R_1 and R_2 are the inner and outer radii of the toroid, h is the height of the toroid, and $\mu_0 = 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ is the permeability of free space.

14.3 Energy in a Magnetic Field

- The energy stored in an inductor U is

$$U = \frac{1}{2}LI^2.$$

- The self-inductance per unit length of coaxial cable is

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{R_2}{R_1}.$$

14.4 RL Circuits

- When a series connection of a resistor and an inductor—an *RL* circuit—is connected to a voltage source, the time variation of the current is
  $I(t) = \frac{\varepsilon}{R}(1 - e^{-Rt/L}) = \frac{\varepsilon}{R}(1 - e^{-t/\tau_L})$ (turning on),
  where the initial current is $I_0 = \varepsilon/R$.
- The characteristic time constant τ is $\tau_L = L/R$, where *L* is the inductance and *R* is the resistance.
- In the first time constant τ, the current rises from zero to $0.632I_0$, and to 0.632 of the remainder in every subsequent time interval τ.
- When the inductor is shorted through a resistor, current decreases as
  $I(t) = \frac{\varepsilon}{R}e^{-t/\tau_L}$ (turning off).
  Current falls to $0.368I_0$ in the first time interval τ, and to 0.368 of the remainder toward zero in each subsequent time τ.

14.5 Oscillations in an LC Circuit

- The energy transferred in an oscillatory manner between the capacitor and inductor in an *LC* circuit occurs at an angular frequency $\omega = \sqrt{\frac{1}{LC}}$.
- The charge and current in the circuit are given by

$$\begin{aligned} q(t) &= q_0 \cos(\omega t + \phi), \\ i(t) &= -\omega q_0 \sin(\omega t + \phi). \end{aligned}$$

14.6 RLC Series Circuits

- The underdamped solution for the capacitor charge in an *RLC* circuit is

$$q(t) = q_0 e^{-Rt/2L} \cos(\omega' t + \phi).$$

- The angular frequency given in the underdamped solution for the *RLC* circuit is

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

CONCEPTUAL QUESTIONS

14.1 Mutual Inductance

1. Show that $N\Phi_m/I$ and $\varepsilon/(dI/dt)$, which are both expressions for self-inductance, have the same units.

2. A 10-H inductor carries a current of 20 A. Describe how a 50-V emf can be induced across it.

3. The ignition circuit of an automobile is powered by a 12-V battery. How are we able to generate large voltages with this power source?

4. When the current through a large inductor is interrupted with a switch, an arc appears across the open terminals of the switch. Explain.

14.2 Self-Inductance and Inductors

5. Does self-inductance depend on the value of the magnetic flux? Does it depend on the current through the wire? Correlate your answers with the equation $N\Phi_m = LI$.

6. Would the self-inductance of a 1.0 m long, tightly wound solenoid differ from the self-inductance per meter of an infinite, but otherwise identical, solenoid?

7. Discuss how you might determine the self-inductance per unit length of a long, straight wire.

8. The self-inductance of a coil is zero if there is no current passing through the windings. True or false?

9. How does the self-inductance per unit length near the center of a solenoid (away from the ends) compare with its value near the end of the solenoid?

14.3 Energy in a Magnetic Field

10. Show that $LI^2/2$ has units of energy.

14.4 RL Circuits

11. Use Lenz's law to explain why the initial current in the *RL* circuit of **Figure 14.12**(b) is zero.

12. When the current in the *RL* circuit of **Figure 14.12**(b) reaches its final value ε/R, what is the voltage across the inductor? Across the resistor?

13. Does the time required for the current in an *RL* circuit to reach any fraction of its steady-state value depend on the emf of the battery?

14. An inductor is connected across the terminals of a battery. Does the current that eventually flows through the inductor depend on the internal resistance of the battery? Does the time required for the current to reach its final value depend on this resistance?

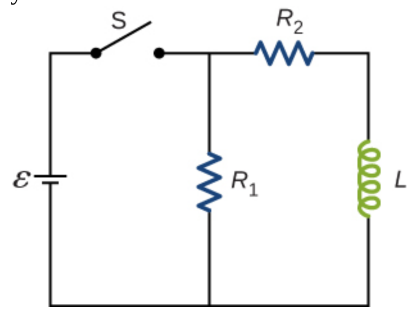
15. At what time is the voltage across the inductor of the *RL* circuit of **Figure 14.12**(b) a maximum?

16. In the simple *RL* circuit of **Figure 14.12**(b), can the emf induced across the inductor ever be greater than the emf of the battery used to produce the current?

17. If the emf of the battery of **Figure 14.12**(b) is reduced by a factor of 2, by how much does the steady-state energy stored in the magnetic field of the inductor change?

18. A steady current flows through a circuit with a large inductive time constant. When a switch in the circuit is opened, a large spark occurs across the terminals of the switch. Explain.

19. Describe how the currents through R_1 and R_2 shown below vary with time after switch S is closed.



20. Discuss possible practical applications of *RL* circuits.

14.5 Oscillations in an LC Circuit

21. Do Kirchhoff's rules apply to circuits that contain inductors and capacitors?

22. Can a circuit element have both capacitance and inductance?

23. In an *LC* circuit, what determines the frequency and the amplitude of the energy oscillations in either the inductor or capacitor?

14.6 RLC Series Circuits

24. When a wire is connected between the two ends of a solenoid, the resulting circuit can oscillate like an *RLC* circuit. Describe what causes the capacitance in this circuit.

25. Describe what effect the resistance of the connecting wires has on an oscillating *LC* circuit.

26. Suppose you wanted to design an *LC* circuit with a frequency of 0.01 Hz. What problems might you encounter?

27. A radio receiver uses an *RLC* circuit to pick out particular frequencies to listen to in your house or car without hearing other unwanted frequencies. How would someone design such a circuit?

PROBLEMS

14.1 Mutual Inductance

28. When the current in one coil changes at a rate of 5.6 A/s, an emf of 6.3×10^{-3} V is induced in a second, nearby coil. What is the mutual inductance of the two coils?

29. An emf of 9.7×10^{-3} V is induced in a coil while the current in a nearby coil is decreasing at a rate of 2.7 A/s. What is the mutual inductance of the two coils?

30. Two coils close to each other have a mutual inductance of 32 mH. If the current in one coil decays according to

$I = I_0 e^{-\alpha t}$, where $I_0 = 5.0\text{ A}$ and $\alpha = 2.0 \times 10^3\ \text{s}^{-1}$, what is the emf induced in the second coil immediately after the current starts to decay? At $t = 1.0 \times 10^{-3}\ \text{s}$?

31. A coil of 40 turns is wrapped around a long solenoid of cross-sectional area $7.5 \times 10^{-3}\ \text{m}^2$. The solenoid is 0.50 m long and has 500 turns. (a) What is the mutual inductance of this system? (b) The outer coil is replaced by a coil of 40 turns whose radius is three times that of the solenoid. What is the mutual inductance of this configuration?

32. A 600-turn solenoid is 0.55 m long and 4.2 cm in diameter. Inside the solenoid, a small $(1.1\text{ cm} \times 1.4\text{ cm})$, single-turn rectangular coil is fixed in place with its face perpendicular to the long axis of the solenoid. What is the mutual inductance of this system?

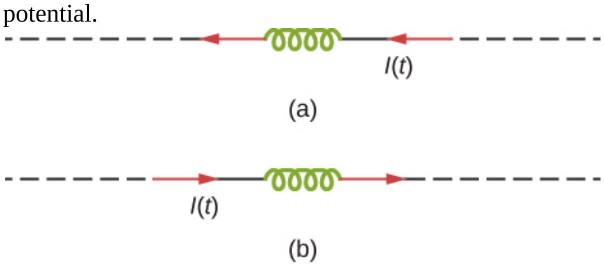
33. A toroidal coil has a mean radius of 16 cm and a cross-sectional area of 0.25 cm^2; it is wound uniformly with 1000 turns. A second toroidal coil of 750 turns is wound uniformly over the first coil. Ignoring the variation of the magnetic field within a toroid, determine the mutual inductance of the two coils.

34. A solenoid of N_1 turns has length l_1 and radius R_1, and a second smaller solenoid of N_2 turns has length l_2 and radius R_2. The smaller solenoid is placed completely inside the larger solenoid so that their long axes coincide. What is the mutual inductance of the two solenoids?

14.2 Self-Inductance and Inductors

35. An emf of 0.40 V is induced across a coil when the current through it changes uniformly from 0.10 to 0.60 A in 0.30 s. What is the self-inductance of the coil?

36. The current shown in part (a) below is increasing, whereas that shown in part (b) is decreasing. In each case, determine which end of the inductor is at the higher potential.



37. What is the rate at which the current though a 0.30-H coil is changing if an emf of 0.12 V is induced across the coil?

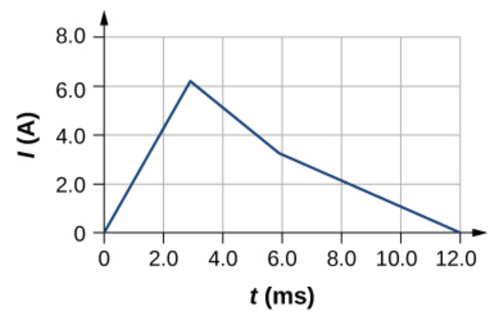
38. When a camera uses a flash, a fully charged capacitor discharges through an inductor. In what time must the 0.100-A current through a 2.00-mH inductor be switched on or off to induce a 500-V emf?

39. A coil with a self-inductance of 2.0 H carries a current that varies with time according to $I(t) = (2.0\text{ A})\sin 120\pi t$. Find an expression for the emf induced in the coil.

40. A solenoid 50 cm long is wound with 500 turns of wire. The cross-sectional area of the coil is 2.0 cm^2 What is the self-inductance of the solenoid?

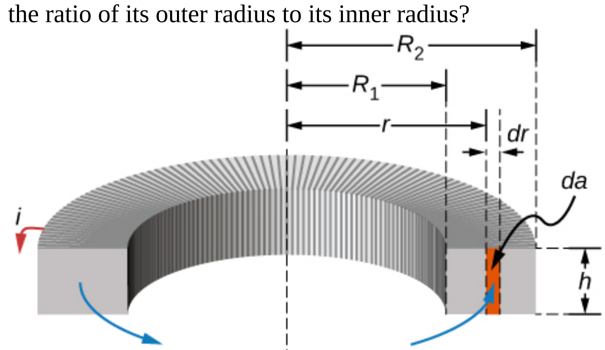
41. A coil with a self-inductance of 3.0 H carries a current that decreases at a uniform rate $dI/dt = -0.050\text{ A/s}$. What is the emf induced in the coil? Describe the polarity of the induced emf.

42. The current *I(t)* through a 5.0-mH inductor varies with time, as shown below. The resistance of the inductor is $5.0\,\Omega$. Calculate the voltage across the inductor at $t = 2.0\text{ ms}$, $t = 4.0\text{ ms}$, and $t = 8.0\text{ ms}$.



43. A long, cylindrical solenoid with 100 turns per centimeter has a radius of 1.5 cm. (a) Neglecting end effects, what is the self-inductance per unit length of the solenoid? (b) If the current through the solenoid changes at the rate 5.0 A/s, what is the emf induced per unit length?

44. Suppose that a rectangular toroid has 2000 windings and a self-inductance of 0.040 H. If $h = 0.10\text{ m}$, what is the ratio of its outer radius to its inner radius?



45. What is the self-inductance per meter of a coaxial cable whose inner radius is 0.50 mm and whose outer radius is 4.00 mm?

14.3 Energy in a Magnetic Field

46. At the instant a current of 0.20 A is flowing through a coil of wire, the energy stored in its magnetic field is 6.0×10^{-3} J. What is the self-inductance of the coil?

47. Suppose that a rectangular toroid has 2000 windings and a self-inductance of 0.040 H. If $h = 0.10\,\text{m}$, what is the current flowing through a rectangular toroid when the energy in its magnetic field is 2.0×10^{-6} J?

48. Solenoid *A* is tightly wound while solenoid *B* has windings that are evenly spaced with a gap equal to the diameter of the wire. The solenoids are otherwise identical. Determine the ratio of the energies stored per unit length of these solenoids when the same current flows through each.

49. A 10-H inductor carries a current of 20 A. How much ice at $0°\,\text{C}$ could be melted by the energy stored in the magnetic field of the inductor? (*Hint*: Use the value $L_\text{f} = 334\,\text{J/g}$ for ice.)

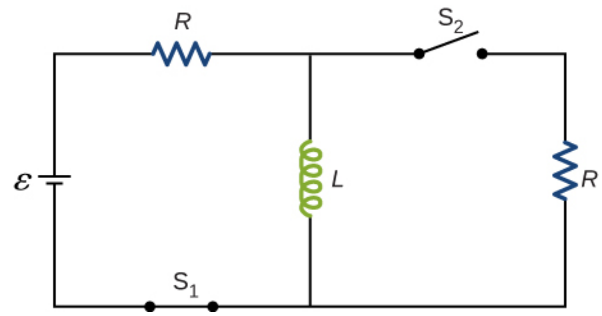
50. A coil with a self-inductance of 3.0 H and a resistance of $100\,\Omega$ carries a steady current of 2.0 A. (a) What is the energy stored in the magnetic field of the coil? (b) What is the energy per second dissipated in the resistance of the coil?

51. A current of 1.2 A is flowing in a coaxial cable whose outer radius is five times its inner radius. What is the magnetic field energy stored in a 3.0-m length of the cable?

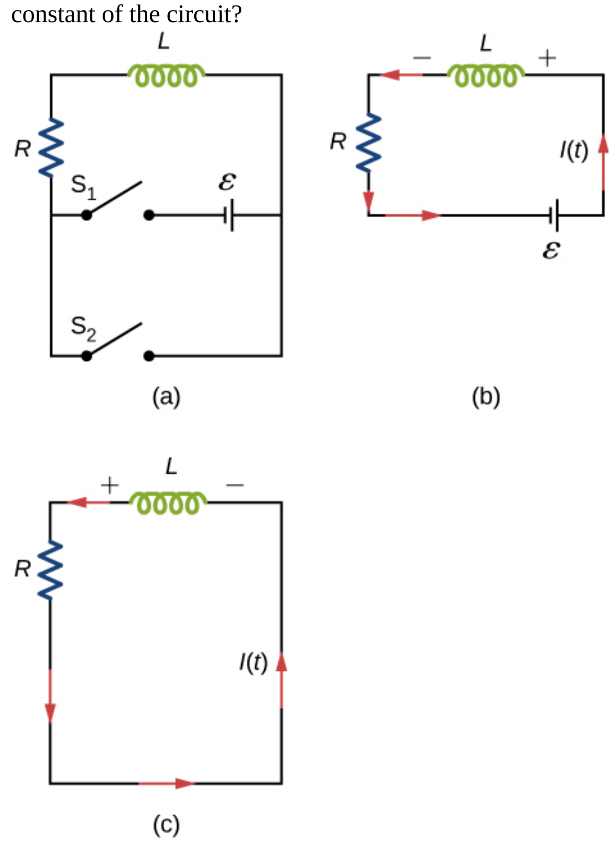
14.4 RL Circuits

52. In **Figure 14.12**, $\varepsilon = 12\,\text{V}$, $L = 20\,\text{mH}$, and $R = 5.0\,\Omega$. Determine (a) the time constant of the circuit, (b) the initial current through the resistor, (c) the final current through the resistor, (d) the current through the resistor when $t = 2\tau_L$, and (e) the voltages across the inductor and the resistor when $t = 2\tau_L$.

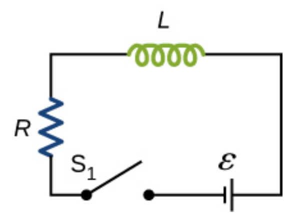
53. For the circuit shown below, $\varepsilon = 20\,\text{V}$, $L = 4.0\,\text{mH}$, and $R = 5.0\,\Omega$. After steady state is reached with S_1 closed and S_2 open, S_2 is closed and immediately thereafter $(\text{at}\ t = 0)$ S_1 is opened. Determine (a) the current through L at $t = 0$, (b) the current through L at $t = 4.0 \times 10^{-4}$ s, and (c) the voltages across L and R at $t = 4.0 \times 10^{-4}$ s.



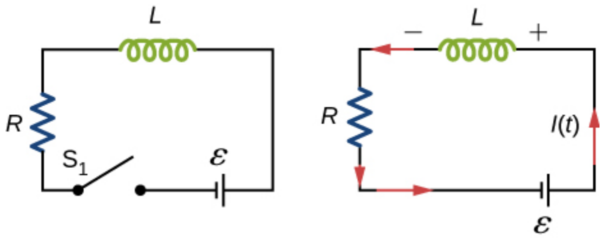
54. The current in the *RL* circuit shown here increases to 40% of its steady-state value in 2.0 s. What is the time constant of the circuit?



55. How long after switch S_1 is thrown does it take the current in the circuit shown to reach half its maximum value? Express your answer in terms of the time constant of the circuit.

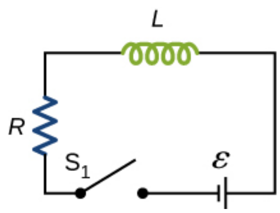


56. Examine the circuit shown below in part (a). Determine *dI/dt* at the instant after the switch is thrown in the circuit of (a), thereby producing the circuit of (b). Show that if *I* were to continue to increase at this initial rate, it would reach its maximum ε/R in one time constant.

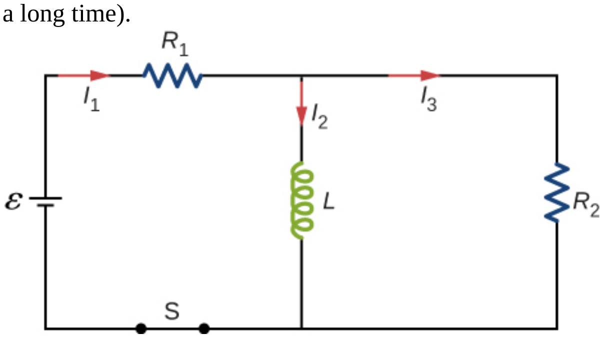


(a) (b)

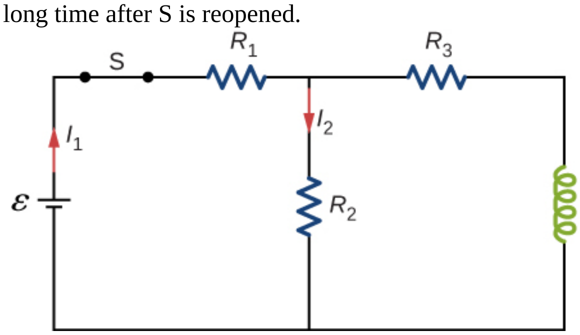
57. The current in the *RL* circuit shown below reaches half its maximum value in 1.75 ms after the switch S_1 is thrown. Determine (a) the time constant of the circuit and (b) the resistance of the circuit if $L = 250\ \text{mH}$.



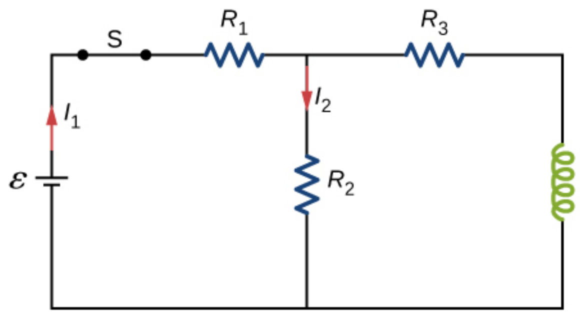
58. Consider the circuit shown below. Find I_1, I_2, and I_3 when (a) the switch S is first closed, (b) after the currents have reached steady-state values, and (c) at the instant the switch is reopened (after being closed for a long time).



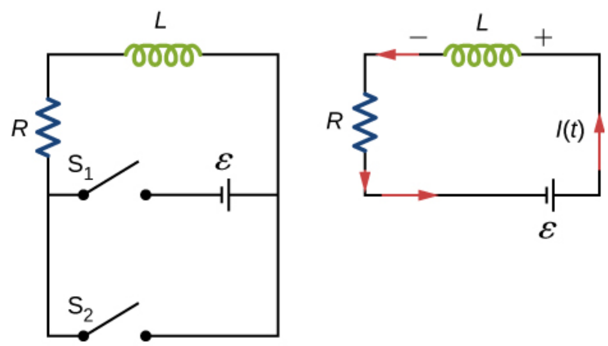
59. For the circuit shown below, $\varepsilon = 50\ \text{V}$, $R_1 = 10\ \Omega$, and $L = 2.0\ \text{mH}$. Find the values of I_1 and I_2 (a) immediately after switch S is closed, (b) a long time after S is closed, (c) immediately after S is reopened, and (d) a long time after S is reopened.



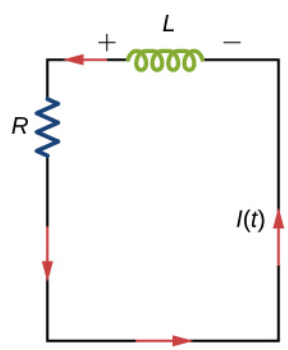
60. For the circuit shown below, find the current through the inductor 2.0×10^{-5} s after the switch is reopened.



61. Show that for the circuit shown below, the initial energy stored in the inductor, $LI^2(0)/2$, is equal to the total energy eventually dissipated in the resistor, $\int_0^\infty I^2(t)Rdt$.



(a) (b)



(c)

14.5 Oscillations in an LC Circuit

62. A 5000-pF capacitor is charged to 100 V and then quickly connected to an 80-mH inductor. Determine (a) the maximum energy stored in the magnetic field of the inductor, (b) the peak value of the current, and (c) the frequency of oscillation of the circuit.

63. The self-inductance and capacitance of an *LC* circuit are 0.20 mH and 5.0 pF. What is the angular frequency at which the circuit oscillates?

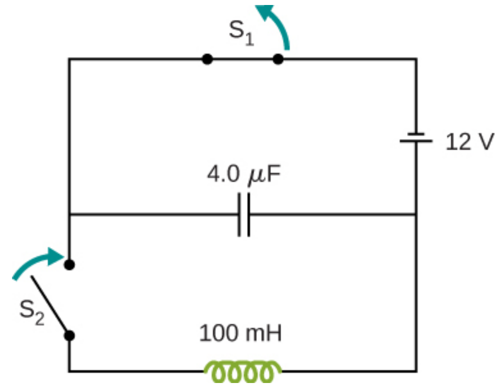
64. What is the self-inductance of an LC circuit that oscillates at 60 Hz when the capacitance is $10\ \mu\text{F}$?

65. In an oscillating LC circuit, the maximum charge on the capacitor is $2.0 \times 10^{-6}\ \text{C}$ and the maximum current through the inductor is 8.0 mA. (a) What is the period of the oscillations? (b) How much time elapses between an instant when the capacitor is uncharged and the next instant when it is fully charged?

66. The self-inductance and capacitance of an oscillating LC circuit are $L = 20\ \text{mH}$ and $C = 1.0\ \mu\text{F}$, respectively. (a) What is the frequency of the oscillations? (b) If the maximum potential difference between the plates of the capacitor is 50 V, what is the maximum current in the circuit?

67. In an oscillating LC circuit, the maximum charge on the capacitor is q_m. Determine the charge on the capacitor and the current through the inductor when energy is shared equally between the electric and magnetic fields. Express your answer in terms of q_m, L, and C.

68. In the circuit shown below, S_1 is opened and S_2 is closed simultaneously. Determine (a) the frequency of the resulting oscillations, (b) the maximum charge on the capacitor, (c) the maximum current through the inductor, and (d) the electromagnetic energy of the oscillating circuit.



69. An LC circuit in an AM tuner (in a car stereo) uses a coil with an inductance of 2.5 mH and a variable capacitor. If the natural frequency of the circuit is to be adjustable over the range 540 to 1600 kHz (the AM broadcast band), what range of capacitance is required?

14.6 RLC Series Circuits

70. In an oscillating RLC circuit, $R = 5.0\ \Omega$, $L = 5.0\ \text{mH}$, and $C = 500\ \mu\text{F}$. What is the angular frequency of the oscillations?

71. In an oscillating RLC circuit with $L = 10\ \text{mH}$, $C = 1.5\ \mu\text{F}$, and $R = 2.0\ \Omega$, how much time elapses before the amplitude of the oscillations drops to half its initial value?

72. What resistance R must be connected in series with a 200-mH inductor of the resulting RLC oscillating circuit is to decay to 50% of its initial value of charge in 50 cycles? To 0.10% of its initial value in 50 cycles?

ADDITIONAL PROBLEMS

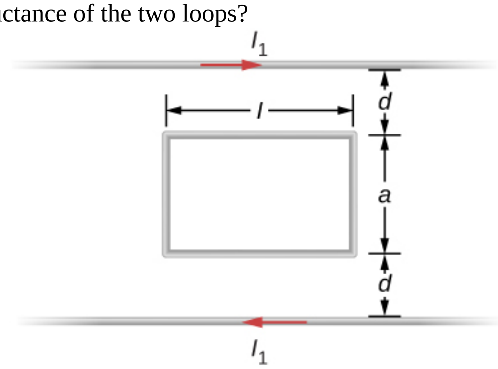
73. Show that the self-inductance per unit length of an infinite, straight, thin wire is infinite.

74. Two long, parallel wires carry equal currents in opposite directions. The radius of each wire is a, and the distance between the centers of the wires is d. Show that if the magnetic flux within the wires themselves can be ignored, the self-inductance of a length l of such a pair of wires is

$$L = \frac{\mu_0 l}{\pi} \ln \frac{d - a}{a}.$$

(*Hint*: Calculate the magnetic flux through a rectangle of length l between the wires and then use $L = N\Phi/I$.)

75. A small, rectangular single loop of wire with dimensions l, and a is placed, as shown below, in the plane of a much larger, rectangular single loop of wire. The two short sides of the larger loop are so far from the smaller loop that their magnetic fields over the smaller fields over the smaller loop can be ignored. What is the mutual inductance of the two loops?



76. Suppose that a cylindrical solenoid is wrapped around a core of iron whose magnetic susceptibility is ***x***. Using **Equation 14.9**, show that the self-inductance of the solenoid is given by

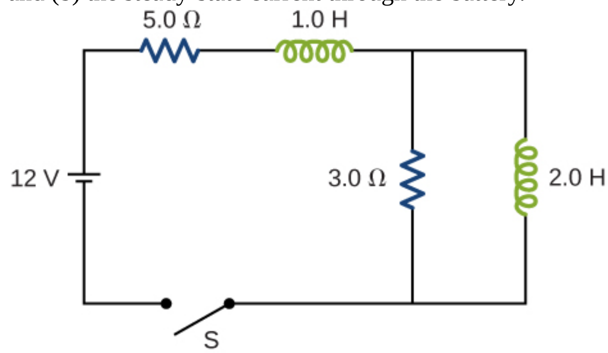
$$L = \frac{(1+x)\mu_0 N^2 A}{l},$$

where l is its length, A its cross-sectional area, and N its total number of turns.

77. A solenoid with $4 \text{ x } 10^7$ turns/m has an iron core placed in it whose magnetic susceptibility is 4.0×10^3. (a) If a current of 2.0 A flows through the solenoid, what is the magnetic field in the iron core? (b) What is the effective surface current formed by the aligned atomic current loops in the iron core? (c) What is the self-inductance of the filled solenoid?

78. A rectangular toroid with inner radius $R_1 = 7.0\,\text{cm}$, outer radius $R_2 = 9.0\,\text{cm}$, height $h = 3.0$, and $N = 3000$ turns is filled with an iron core of magnetic susceptibility 5.2×10^3. (a) What is the self-inductance of the toroid? (b) If the current through the toroid is 2.0 A, what is the magnetic field at the center of the core? (c) For this same 2.0-A current, what is the effective surface current formed by the aligned atomic current loops in the iron core?

79. The switch S of the circuit shown below is closed at $t = 0$. Determine (a) the initial current through the battery and (b) the steady-state current through the battery.



80. In an oscillating *RLC* circuit, $R = 7.0\,\Omega$, $L = 10\,\text{mH}$, and $C = 3.0\,\mu\text{F}$. Initially, the capacitor has a charge of $8.0\,\mu\text{C}$ and the current is zero. Calculate the charge on the capacitor (a) five cycles later and (b) 50 cycles later.

81. A 25.0-H inductor has 100 A of current turned off in 1.00 ms. (a) What voltage is induced to oppose this? (b) What is unreasonable about this result? (c) Which assumption or premise is responsible?

CHALLENGE PROBLEMS

82. A coaxial cable has an inner conductor of radius a, and outer thin cylindrical shell of radius b. A current I flows in the inner conductor and returns in the outer conductor. The self-inductance of the structure will depend on how the current in the inner cylinder tends to be distributed. Investigate the following two extreme cases. (a) Let current in the inner conductor be distributed only on the surface and find the self-inductance. (b) Let current in the inner cylinder be distributed uniformly over its cross-section and find the self-inductance. Compare with your results in (a).

83. In a damped oscillating circuit the energy is dissipated in the resistor. The Q-factor is a measure of the persistence of the oscillator against the dissipative loss. (a) Prove that for a lightly damped circuit the energy, U, in the circuit decreases according to the following equation.

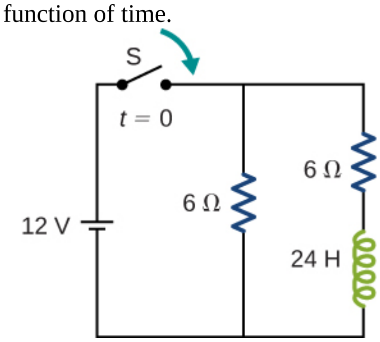
$$\frac{dU}{dt} = -2\beta U, \quad \text{where } \beta = \frac{R}{2L}.$$

(b) Using the definition of the Q-factor as energy divided by the loss over the next cycle, prove that Q-factor of a lightly damped oscillator as defined in this problem is

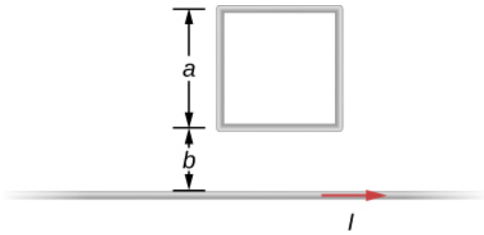
$$Q \equiv \frac{U_{\text{begin}}}{\Delta U_{\text{one cycle}}} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

(*Hint:* For (b), to obtain Q, divide E at the beginning of one cycle by the change ΔE over the next cycle.)

84. The switch in the circuit shown below is closed at $t = 0\,\text{s}$. Find currents through (a) R_1, (b) R_2, and (c) the battery as function of time.



85. A square loop of side 2 cm is placed 1 cm from a long wire carrying a current that varies with time at a constant rate of 3 A/s as shown below. (a) Use Ampère's law and find the magnetic field as a function of time from the current in the wire. (b) Determine the magnetic flux through the loop. (c) If the loop has a resistance of $3\,\Omega$, how much induced current flows in the loop?



86. A rectangular copper ring, of mass 100 g and resistance $0.2\,\Omega$, is in a region of uniform magnetic field that is perpendicular to the area enclosed by the ring and horizontal to Earth's surface. The ring is let go from rest when it is at the edge of the nonzero magnetic field region (see below). (a) Find its speed when the ring just exits the region of uniform magnetic field. (b) If it was let go at $t = 0$, what is the time when it exits the region of magnetic field for the following values: $a = 25\text{ cm},\ b = 50\text{ cm},\ B = 3\text{ T},$ and $g = 9.8\text{ m/s}^2$? Assume the magnetic field of the induced current is negligible compared to 3 T.